

Almaty 2021**LIST OF EXECUTORS**

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| --- | --- | --- | --- |
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**ABSTRACT**

Report 40 pp., 1 book, 27 sources, 2 annexes

LOADED ORDINARY DIFFERENTIAL EQUATION, MULTIPOINT BOUNDARY VALUE PROBLEM, PARAMETRIZATION METHOD, PROBLEM WITH PARAMETER, NUMERICAL METHOD

The object of research is linear multipoint boundary value problems for essentially loaded ordinary differential equations.

The purpose of the research is establishment solvability conditions and development numerical methods for solving multipoint boundary value problems for essentially loaded ordinary differential equations.

The method of parametrization, modern methods of the theory of differential equations and functional analysis are applied.

The following results were obtained:

An algorithm for finding a solution of a linear two-point boundary value problem for essentially loaded ordinary differential equations is constructed;

An algorithm for finding a solution of a linear multipoint boundary value problem for essentially loaded ordinary differential equations is constructed and conditions for its convergence are established;

The unique solvability of linear multipoint boundary value problems for essentially loaded ordinary differential equations is established;

Numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations are developed.

Research results have theoretical significance and can be used in mathematical modeling of problems for loaded differential equations.

Scientific publications. According to the results of the research from the beginging of October 2020 there have been published 5 scientific articles, including:

* 1 article in international rating journal included in the Scopus database;
* 2 articles in international rating journals (included in the database of Web of Science Clarivate Analytics, Emerging Sources Citation Index) without impact factor;
* 2 articles in domestic journals recommended by CQAEA of MES of RK.

**РЕФЕРАТ**

Есеп 40 б., 1 кітап, 27 әдебиет көздері, 2 қос.

ЖҮКТЕЛГЕН ЖӘЙ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ, КӨП НҮКТЕЛІ ШЕТТІК ЕСЕП, ПАРАМЕТРЛЕУ ӘДІСІ, ПАРАМЕТРІ БАР ЕСЕП, САНДЫҚ ӘДІС

Зерттеу нысаны елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықты көп нүктелі шеттік есептер болып табылады.

Зерттеу мақсаты - елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін көп нүктелі шеттік есептерді шешудің сандық әдістерін әзірлеу және шешілімділік шарттарын тағайындау.

Параметрлеу әдісі, дифференциалдық теңдеулер мен функционалдық анализдің қазіргі әдістері қолданылған.

Келесі нәтижелер алынды:

Елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықты екі нүктелі шеттік есептердің шешімін табу алгоритмі құрылды;

Елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықты көп нүктелі шеттік есептердің шешімін табу алгоритмі құрылды және оның жинақтылық шарттары тағайындалды;

Елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықты көп нүктелі шеттік есептердің бірмәнді шешілімділігі тағайындалды;

Елеулі түрде жүктелген жәй дифференциалдық теңдеулер үшін сызықты көп нүктелі шеттік есептерді шешудің сандық әдістері жасалды.

Зерттеу нәтижелерінің теориялық маңызы бар және жүктелген дифференциалдық теңдеулер үшін қолданбалы есептерді математикалық моделдеу кезінде пайдаланылуы мүмкін.

Ғылыми басылымдар. Зерттеу нәтижелері бойынша 2020 жылдың қазан айынан басынан бастап жоба қызметкерлері 5 ғылыми мақалалар жариялады, оның ішінде:

* 1 мақала (Article) Scopus ДБ кіретін халықаралық рейтингтік журналда;
* 2 мақала импакт-факторсыз халықаралық рейтингтік журналдарда (WEB of Science Clarivate Analytics, Emerging Sources Citation Index ДҚ-ға кіретін);
* 2 мақала ҚР БҒМ БҒССҚК ұсынған қазақстандық журналдарда.

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**INTRODUCTION**

The report contains research on the theory of multipoint boundary value problems for essentially loaded ordinary differential equations.

Mathematical description of various dynamic processes control, in which the future course of processes depends not only on the present, but also is significantly determined by the prehistory of the process, is carried out using ordinary differential equations with memory of various types, also called equations with aftereffect or loaded differential equations. Loaded differential equations are used in solving the problems of long-term forecasting and regulation of the level of groundwater and soil moisture. Loaded differential equations have a number of features that must be taken into account when setting problems for these equations and creating methods for their solutions. One of the peculiarities of loaded differential equations is that such equations can be unsolvable without additional conditions.

When studying a moving observation point in feedback devices, essentially loaded differential equations often appear, where the order of the derivative in the loaded term is equal to or higher than the order of the differential part of the equation. In contrast to the previously studied loaded differential equations, the loaded term in the equation will not be a certain perturbation of its differential part.

Despite the large number of papers devoted to the study and solution of boundary value problems for loaded differential equations, many questions related to the solvability of boundary value problems for essentially loaded ordinary differential equations remain unsolved. Expansion of the spheres of application of essentially loaded differential equations, and the existence of unsolved problems lead to the need to build new effective methods of studying and solving problems for these equations.

Methods for finding approximate solutions of linear multipoint boundary value problems for essentially loaded ordinary differential equations are proposed and conditions for the existence of the solution in terms of the initial data are established in this paper.

The fundamental difference between the ideas and the scientific significance of the results of the work from existing analogues lies in the construction of effective algorithms for finding their solutions.

The applied methodology for scientific research is assessed by high quality: the parameterization method and constructive methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations are used.

The set problems in the project for 2020-2021 fully implemented.

The strengths of the project include: it uses methods developed by the project leader and scientists from Kazakhstan; scientific results on the conducted research are published in journals included in the list of recommended publications of the Committee for Quality Assurance in the sphere of Education and Science of the MES RK and in a peer-reviewed scientific publication with a CiteScore percentile in the Scopus database not less than 50 (fifty).

The conducted research and results correspond to the work plan of research works on the topic No. AP08955489: "Methods of solving multipoint boundary value problems for systems of loaded differential equations" for 2020-2022 with a period of implementation of 12 months Specialized scientific direction 8.1: "Fundamental and applied research in mathematics and mechanics" Priority 8: "Research in the field of natural sciences".

Researches on the topic "Methods of solving multipoint boundary value problems for systems of loaded differential equations" for 2020-2021 are reflected in this report. The obtained results are a further development of the theory of boundary value problems for significantly loaded ordinary differential equations.

For 2020 of reporting period 2020-2022 with a period of implementation of 12 months on the topic, an interim report on scientific-research work "Methods of solving multipoint boundary value problems for systems of loaded differential equations", inv №0220РК01618 was prepared.

**MAIN PART OF THE RESEARCH WORK REPORT**

**1 An algorithm for finding a solution and solvability conditions of two-point boundary value problems for essentially loaded ordinary differential equations**

Loaded differential equations are used in solving the problems of long-term forecasting and regulation of the level of groundwater and soil moisture [1-4]. Note that loaded differential equations in the literature are also called boundary differential equations [5, 6]. Also, a loaded differential equation was called a differential equation that includes the values of the desired function and its derivatives at fixed points in the domain [7]. Various problems for loaded differential equations and methods for finding their solutions are considered in [8-26].

This section is devoted to a linear two-point boundary value problem for essentially loaded differential equations. Using the properties of an essentially loaded differential equation, the considered problem is reduced to a two-point boundary value problem for loaded differential equations. This problem is investigated by the parametrization method [27]. Algorithms for finding solutions to boundary value problems for loaded differential equations are constructed and conditions for their feasibility are obtained.

The results of the section correspond to paragraph 1.1. of Calendar plan for 2020.

On for essentially loaded ordinary differential equations:

(1.1)

we consider a linear two-point boundary value problem with the condition

(1.2)

where the -matriсes , , (), and -vector-function are continuous on , and are constant - matrices, is constant -vector, and , .

Let denote the space of continuous on functions with norm .

A solution to problem (1.1), (1.2) is a continuously differentiable on function satisfying the essentially loaded differential equations (1.1) and boundary condition (1.2).

The value of the derivative at the loading point can be found from the system of differential equations (1.1). Using equation (1.1), we define

Then

(1.3)

Assume that the matrix is invertible. We obtain

(1.4)

We consider the following linear two-point boundary value problem for loaded differential equations

(1.5)

(1.6)

where

Let us consider an example showing that loads influences significantly to the property of boundary value problem. Consider the following Cauchy problem for the loaded differential equation

(1.7)

(1.8)

Integrating equation (1.7), using the initial condition (1.8), we have

Since the value satisfies the equality

But if we take then the equation (1.9) does not hold and the Cauchy problem (1.7), (1.8) is not solved. At the same time, the Cauchy problem for a linear system of ordinary differential equations (without loading) always has a unique solution.

On we consider a periodic boundary value problem for an ordinary differential equation

The General solution of the differential equation has the form: Substituting the General solution in the boundary conditions for determining , we obtain the relation: . Since there is no such number , the problem has no solution.

Now, adding the load at the point to the right side of the differential equation we obtain the following periodic boundary value problem for a loaded differential equation

and the solution of this problem has the form

Scheme of parametrization method.

In this paper, boundary value problem (1.5), (1.6) is investigated by the parametrization method. The interval is divided into subintervals by loading points:

.

Introduce as a spase of systems of functions , where are continuous on and have finite left-sided limits for all , with norm .

Let be the restriction of the function to the th interval , i.e. for , and we reduce problem (1.5), (1.6) to the equivalent multipoint boundary value problem

(1.10)

(1.11)

(1.12)

where (1.12) are conditions for matching the solution at the interior points of partition.

The solution of problem (1.10) - (1.12) is a system of functions , where the functions are continuously differentiable on , which satisfies system of loaded differential equations (1.10) and conditions (1.11), (1.12).

Problems (1.5), (1.6) and (1.10)-(1.12) are equivalent. If is a solution of problem (1.5), (1.6), then the system of functions , where and is a solution of problem (1.10)-(1.12). Conversely, if a system of vector-functions is a solution of problem (1.10)-(1.12), then the function defined by the equalities , is a solution of the original problem (1.5), (1.6).

Let us introduce the notation and perform a replacement of the function on each -th interval . Then problem (1.10) - (1.12) is reduced to an equivalent multipoint boundary value problem for differential equations with parameters

(1.13) (1.14)

(1.15)

(1.16)

A pair with elements , is said to be a solution to problem (1.13)-(1.16) if the functions are continuously differentiable on and satisfies the system of ordinary differential equations (1.13) with and conditions (1.14) – (1.16).

Problems (1.5), (1.6) and (1.13)-(1.16) are equivalent. If a pair with elements , , is a solution of (1.13)-(1.16) , then the function defined by the equalities ,, will be the solution of the original problem (1.5), (1.6). Conversely, if the is a solution of problem (1.5), (1.6), then the pair where , and , is a solution of problem (1.13)-(1.16).

However, the problem (1.13) - (1.16) differs from the problem (1.10) - (1.12) in that here the initial conditions appeared at the points which allow, for fixed to determine the functions from the Volterra integral equations of the second kind

In the equation (1.17), replacing by the corresponding right-hand side and then repeating this process times, we obtain

Introducing the notations

we obtain a representation of the function of the form

(1.18)

From (1.18) we find

Substituting the corresponding right-hand sides of (1.18) into conditions (1.15), (1.16), we obtain the system of equations for the unknown parameters

(1.19)

(1.20)

where is an identity matrix of dimension Denote the matrix corresponding to the left-hand side of system (1.19), (1.20) by and introduce the vectors

=,

we write it in the form

Thus, we have the closed system (1.17), (1.21) to find the unknown pair a solution to problem (1.13) - (1.16). A solution to problem (1.13) - (1.16) the pair is found as the limit of the sequence of pairs determined by the following algorithm:

Step 0: We assume that for the chosen matrix is invertible and we find initial approximation with respect to the parameter from the equation i.e.

b) Using the components of vector and solving the Cauchy problems (1.13), (1.14) at on the intervals , we obtain the functions .

Step 1: Substituting the found into the right-hand side of (1.21), from the equation we determine .

b) On the intervals solving the Cauchy problems (1.13), (1.14) at , we find the functions , and so on.

Continuing the process, at the -th step we obtain a system of pairs Note that in part b) for fixed values of the parameter , the solution to the Cauchy problem is found separately on each interval .

Convergence conditions of the algorithm and the unique solvability of the boundary value problem.

Theorem 1. Suppose that for some the matrix is invertible, and the following inequalities are true:

where ,

Then the linear two-point boundary value problem for loaded differential equations (1.5), (1.6) has a unique solution.

**2 An algorithm for finding a solution of linear multipoint boundary value problems for essentially loaded ordinary differential equations**

A linear multipoint boundary value problem for essentially loaded ordinary differential equations is investigated. Assuming the invertibility of the matrix composed by the coefficients at the values ​​of the derivative of the desired function at the load points, the investigating problem is reduced to a multipoint boundary value problem for a loaded ordinary differential equation. By splitting the interval and introducing additional parameters, the linear multipoint boundary value problem for loaded ordinary differential equations is reduced to an equivalent boundary value problem with a parameter. An equivalent boundary value problem with parameters consists of the Cauchy problem for a system of ordinary differential equations with parameters, a multipoint condition, and a continuity condition. An algorithm for finding a solution to a multipoint boundary value problem for a loaded ordinary differential equation is proposed.

The results of the section correspond to paragraph 2.1. of Calendar plan for 2021.

A linear multipoint boundary value problem for the essentially loaded ordinary differential equations is considered on

(2.1)

(2.2)

where the -matriсes , (), and -vector-function are continuous on , () - are constant - matrices, is constant -vector, and , .

A solution to problem (2.1), (2.2) is a continuously differentiable on function satisfying the essentially loaded differential equations (2.1) and boundary condition (2.2).

The value of the derivatives at the loading points can be found from the system of differential equations (2.1). Using equation (2.1), we define :

(2.3)

We rewrite (2.3) in the following form

(2.4)

Here i.е.

where identity matrix of dimension .

We assume that the matrix is invertible. The inverse matrix is denoted by , i.е. where Then from (2.4) we can uniquely determine Thus, the components of the vector allow us to find the values of the derivatives at the points .

Consider the following linear multipoint boundary value problem for loaded differential equations

(2.5)

(2.6)

where

Boundary value problem (2.5), (2.6) is investigated by the parametrization method. The interval is divided into subintervals by loading points: .

Introduce as a spase of systems of functions , where are continuous on and have finite left-sided limits for all , with norm .

Let be the restriction of the function to the th interval , i.e. for , and introducing the notation and perform a replacement of the function on each -th interval . Then problem (2.5), (2.6) is reduced to an equivalent multipoint boundary value problem for differential equations with parameters

(2.7) (2.8)

(2.9)

(2.10)

A pair with elements , is said to be a solution to problem (2.7) - (2.10) if the functions are continuously differentiable on and satisfies the system of ordinary differential equations (2.7) with and conditions (2.8) – (2.10).

Problems (2.5), (2.6) and (2.7) - (2.10) are equivalent. If a pair with elements , , is a solution of (2.7) - (2.10), then the function defined by the equalities ,, will be the solution of the original problem (2.5), (2.6). Conversely, if the is a solution of problem (2.5), (2.6), then the pair where , and , is a solution of problem (2.7) - (2.10).

In problem (2.7) - (2.10) initial conditions appeared at the points which allow, for fixed to determine the functions from the Volterra integral equations of the second kind

In the equation (2.11), replacing by the corresponding right-hand side and then repeating this process times, we obtain

(2.12)

where

From (2.12) we find

Substituting the corresponding right-hand sides of (2.12) into conditions (2.9), (2.10), we obtain the system of equations for the unknown parameters

(2.13)

(2.14)

where is an identity matrix of dimension Denote the matrix corresponding to the left-hand side of system (2.13), (2.14) by and introduce the vectors

=,

we write it in the form

A solution to problem (2.7) - (2.10) the pair is found as the limit of the sequence of pairs determined by the following algorithm:

Step 0: We assume that for the chosen matrix is invertible and we find initial approximation with respect to the parameter from the equation i.e.

b) Using the components of vector and solving the Cauchy problems (2.7), (2.8) at on the intervals , we obtain the functions .

Step 1: Substituting the found into the right-hand side of (2.15), from the equation we determine .

b) On the intervals solving the Cauchy problems (2.7), (2.8) at , we find the functions , and so on.

Continuing the process, at the -th step we obtain a system of pairs Note that in part b) for fixed values of the parameter , the solution to the Cauchy problem is found separately on each interval .

**3 Solvability of linear multipoint boundary value problems for essentially loaded ordinary differential equations**

A linear multipoint boundary value problem for essentially loaded ordinary differential equations is investigated. Assuming the invertibility of the matrix composed by the coefficients at the values of the derivative of the desired function at the load points, the investigating problem is reduced to a multipoint boundary value problem for a loaded ordinary differential equation. An algorithm for finding a solution to a multipoint boundary value problem for a loaded ordinary differential equation is proposed and conditions for its convergence are established.

The results of the section correspond to paragraph 2.2. of Calendar plan for 2021.

A linear multipoint boundary value problem for the essentially loaded ordinary differential equations is considered on

(3.1)

(3.2)

where the -matriсes , (), (), and -vector-function are continuous on , () - are constant - matrices, is constant -vector, and , .

A solution to problem (3.1), (3.2) is a continuously differentiable on function satisfying the essentially loaded differential equations (3.1) and boundary condition (3.2).

The value of the derivatives at the loading points can be found from the system of differential equations (3.1). Using equation (3.1), we define :

(3.3)

We rewrite (3.3) in the following form

(3.4)

Here i.е.

where identity matrix of dimension .

We assume that the matrix is invertible. The inverse matrix is denoted by , i.е. where Then from (3.4) we can uniquely determine Thus, the components of the vector allow us to find the values of the derivatives at the points .

Consider the following linear multipoint boundary value problem for loaded differential equations

(3.5)

(3.6)

where

Boundary value problem (3.5), (3.6) is investigated by the parametrization method. The interval is divided into subintervals by loading points: .

Let be the restriction of the function to the th interval , i.e. for , and introducing the notation and perform a replacement of the function on each -th interval . Then problem (3.5), (3.6) is reduced to an equivalent multipoint boundary value problem for differential equations with parameters

(3.7) (3.8)

(3.9)

(3.10)

Since the value of the solution at is on the right-hand side of system (3.7), the number of introduced parameters is one more than the number of unknown functions.

A pair with elements , is said to be a solution to problem (3.7) - (3.10) if the functions are continuously differentiable on and satisfies the system of ordinary differential equations (3.7) with and conditions (3.8) – (3.10).

Problems (3.5), (3.6) and (3.7) - (3.10) are equivalent. If a pair with elements , , is a solution of (3.7) - (3.10), then the function defined by the equalities ,, will be the solution of the original problem (3.5), (3.6). Conversely, if the is a solution of problem (3.5), (3.6), then the pair where , and , is a solution of problem (3.7) - (3.10).

In problem (3.7) - (3.10) initial conditions appeared at the points which allow, for fixed to determine the functions from the Volterra integral equations of the second kind

In the equation (3.11), replacing by the corresponding right-hand side and then repeating this process times, we obtain

(3.12)

where

From (3.12) we find

Substituting the corresponding right-hand sides of (3.12) into conditions (3.9), (3.10), we obtain the system of equations for the unknown parameters

(3.13)

(3.14)

where is an identity matrix of dimension Denote the matrix corresponding to the left-hand side of system (3.13), (3.14) by and introduce the vectors

=,

we write it in the form

A solution to problem (3.7) - (3.10) the pair is found as the limit of the sequence of pairs determined by the following algorithm:

Step 0: We assume that for the chosen matrix is invertible and we find initial approximation with respect to the parameter from the equation i.e.

b) Using the components of vector and solving the Cauchy problems (3.7), (3.8) at on the intervals , we obtain the functions .

Step 1: Substituting the found into the right-hand side of (3.15), from the equation we determine .

b) On the intervals solving the Cauchy problems (3.7), (3.8) at , we find the functions , and so on.

Continuing the process, at the -th step we obtain a system of pairs Note that in part b) for fixed values of the parameter , the solution to the Cauchy problem is found separately on each interval .

Convergence conditions of the algorithm and the unique solvability of the boundary value problem.

Theorem 2. Suppose that for some the matrix is invertible, and the following inequalities are true:

где ,

Then the linear multipoint boundary value problem for loaded differential equations (3.5), (3.6) has a unique solution.

**4 Development of numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations**

In this section, based on the parametrization method and numerical methods, a numerical method for solving a linear multipoint boundary value problem for essentially loaded ordinary differential equations is developed and algorithms for their implementation are proposed. By splitting the interval and introducing additional parameters, the linear multipoint boundary value problem for loaded ordinary differential equations is reduced to an equivalent boundary value problem with a parameter. An equivalent boundary value problem with parameters consists of the Cauchy problem for a system of ordinary differential equations with parameters, a multipoint condition, and a continuity condition. The solution to the Cauchy problem for a system of ordinary differential equations with parameters is constructed using the fundamental matrix of the differential equation. Substituting the values at the corresponding points of the constructed solution into the multipoint condition and the continuity condition, a system of linear algebraic equations for the parameters is compiled. A numerical method is proposed for solving the considered problem, based on the solving of the constructed system and the Runge-Kutta method of the 4th order for solving the Cauchy problem on subintervals.

The results of the section correspond to paragraph 2.3. of Calendar plan for 2021. The introduction of additional parameters allows to obtain the initial data , for the components of the unknown system of the function and problem (3.7), (3.8) for fixed values of the parameters is the Cauchy problem. On the intervals the Cauchy problem is solved separately and the fundamental matrix is used to find the solution.

Using the fundamental matrix of differential equation on the solution of the Cauchy problem (3.7), (3.8) can be written in the form

. (3.16)

Solving the (3.16), we find a representation in terms of and . Substituting the (3.16) into the conditions (3.9) and (3.10) we obtain a system of equations for finding the unknown parameters

(3.17)

(3.18)

We denote the matrix corresponding to the left side of the system of equations (3.17), (3.18) by and write the system in the form

(3.19)

where

It is not difficult to establish that the solvability of the boundary value problem (3.5), (3.6) is equivalent to the solvability of the system (3.19).

Further we consider the Cauchy problems for ordinary differential equations on subintervals

(3.20)

where is either ( matrix, or vector, both continuous on . Consequently, solution to problem (3.20) is a square matrix or a vector of dimension . Denote by the solution to the Cauchy problem (3.20). Obviously,

where is a fundamental matrix of differential equation (3.20) on the r-th interval.

The proposed numerical method is based on the construction and solving of system (3.19). As can be seen from equations (3.17), (3.18), the coefficients and the right-hand side of system (3.19) are found as a solution to the matrix and vector Cauchy problems for ordinary differential equations

(3.21)

(3.22)

(3.23)

Let consider . Divide each r-th interval into parts with step. Assume on each interval the variable takes its discrete values: and denote by the set of such points. We find the approximate values of the coefficients and the right-hand side of system (3.19) by solving the matrix and vector Cauchy problems (3.21) - (3.23) by the Runge-Kutta method of the 4th order of accuracy with a step on each r-th interval. And we find the values of the (-matrices and n-vector on .

Then we obtain the following approximate system of algebraic equations for the parameters :

(3.24)

Solving the system (3.24), we find . As noted above, the elements of =() are the values of approximate solution to problem (3.5), (3.6) in the starting points of subintervals: . The approximate values of the solution at the remaining points of the subintervals are determined by the solving of the Cauchy problems

(3.25)

. (3.26)

To solve the Cauchy problems (3.25), (3.26) on the basis of the Runge-Kutta method of the 4th order of accuracy, we find the numerical solution of the linear multi-point boundary value problem for the system of loaded differential equations (3.5), (3.6). We see that the solution to the boundary value problem (3.5), (3.6) is also a solution to the boundary value problem (3.1), (3.2) when the matrix G (θ) is invertible.

**CONCLUSION**

This report contains research on the topic "Methods of solving multipoint boundary value problems for systems of loaded differential equations" performed in 2020-2021 in the field of boundary value problems for systems of essentially loaded ordinary differential equations.

An algorithm for finding a solution of a linear two-point boundary value problem for essentially loaded ordinary differential equations is constructed.

An algorithm for finding a solution of a linear multipoint boundary value problem for essentially loaded ordinary differential equations is constructed and conditions for its convergence are established.

The unique solvability of linear multipoint boundary value problems for essentially loaded ordinary differential equations is established.

Numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations are developed.

The research work carried out during the reporting period on the topic concerns actual problems of the modern theory of boundary value problems for loaded differential equations dictated by real physical processes and is mainly theoretical in nature. The results obtained by the performers of the project are new and sufficiently fully reflect the content of the tasks.

The high level of the research work performed is characterized by the participation of the project performers in numerous publications in reputable mathematical journals, which is reflected in the List of used sources and Appendix A - in the list of published works of this report. The plan of research activities is given in Appendix B.

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**APPENDIX А**

**List of published works**

List of publications for 2020

In domestic publications included in the Web of Science database

1 Kadirbayeva Zh.M., Dzhumabaev A.D. Numerical implementation of solving a control problem for loaded diﬀerential equations with multi-point condition // Bulletin of the Karaganda university. Mathematics series. 2020. – № 4(100). –P. 81-91. [https://doi.org/10.31489/2020M4/ 81-91](https://doi.org/10.31489/2020M4/%2081-91)

In journals included in the list of recommended publications of the Committee for control in education and science MES RK

1 Kadirbayeva Zh.M., Karakenova S.G. Numerical solution of the multipoint boundary value problems for essentially loaded ordinary differential equations // Kazakh Mathematical Journal. – 2020. – Vol. 20, No. 4. – P.47-57.

2 Bakirova E.A., Minglibayeva B.B., Kasymova A.B. An algorithm for solving multipoint boundary value problem for the loaded differential and Fredholm integro-differential equations // Kazakh Mathematical Journal. – 2020. – Vol. 20, No. 4. – P.107-118.

List of publications for 2021

In publications included in the Web of Science and Scopus databases

1 Kadirbayeva Zh.M. A numerical method for solving boundary value problem for essentially loaded diﬀerential equations // Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 3, pp. 551–559. (Scopus SJR=0.422, Percentile 50 in Mathematics, General Mathematics). <https://doi.org/10.1134/S1995080221030112>

In domestic publications included in the Web of Science database

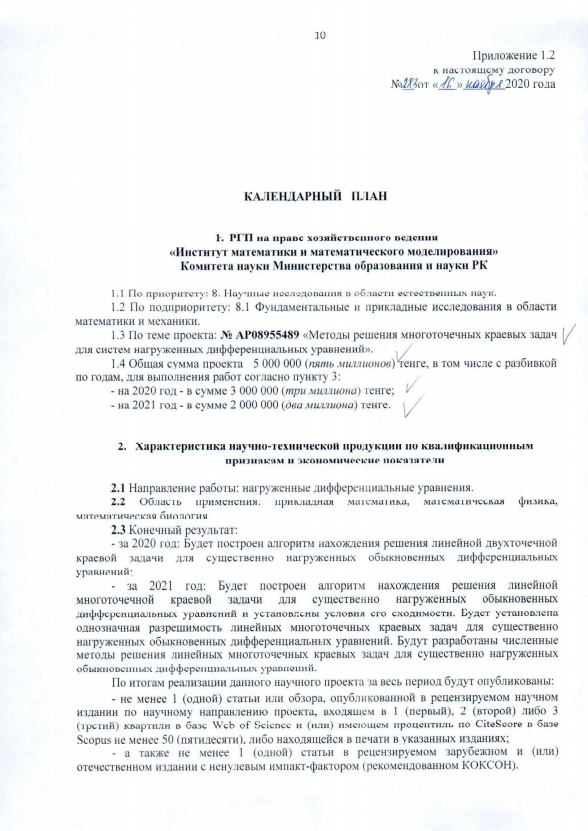
1 Kadirbayeva Zh.M., Bakirova E.A., Dauletbayeva A.Sh., Kassymgali A.A. An algorithm for solving a boundary value problem for essentially loaded differential equations // News of the NAS RK. Phys.-Math. Series. 2021. Volume 2, Number 336. -P.6-14. <https://doi.org/10.32014/2021.2518-1726.15>

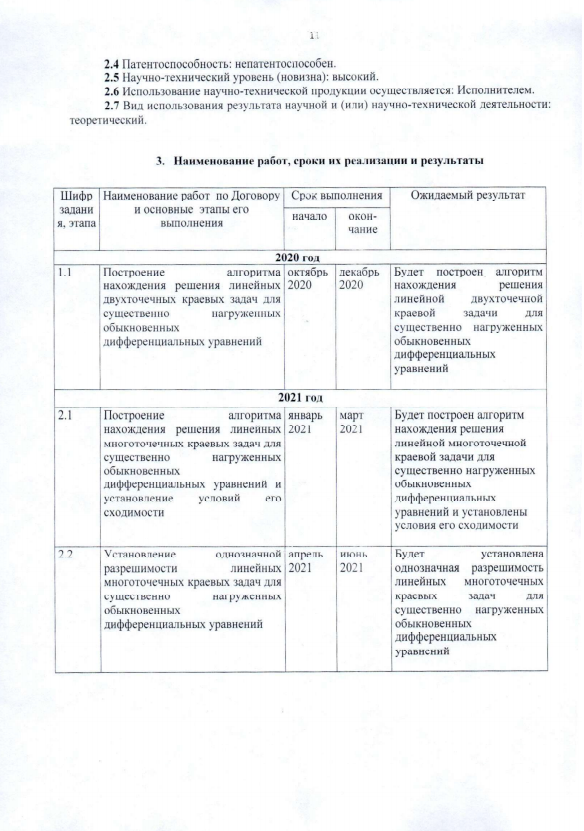
Abstracts and materials of scientific conferences

1 Kadirbayeva Zh.M. A problem for essentially loaded differential equations // Annual International April Mathematical Conference-2021. (5-8 April): - Almaty, 2021 -P. 86-87.

2 Kadirbayeva Zh.M. A numerical algorithm for solving problem for a system of essentially loaded differential equations // 8th European Congress of Mathematics. (20 - 26 June): -Portorož, Slovenia, 2021. –P. 650-651.

**APPENDIX B**

**Calendar plan of works** 





**Translation into English of the calendar plan of the grant project**

**"Methods of solving multipoint boundary value problems for**

**systems of loaded differential equations"**

Appendix 1.2

to Agreement № \_\_ from \_\_\_\_\_\_2020

for grant funding

**WORK SCHEDULE**

Under agreement № 283 from November 16, 2020

1. **RSE on the right of economic management “Institute of mathematics and mathematical modeling” of the Committee of science of the Ministry of education and science of the Republic of Kazakhstan**
   1. By priority: 8 Research in the field of natural sciences
   2. By sub-priority: 8.1 Fundamental and applied research in mathematics and mechanics
   3. On the theme of the project: **No. AP08955489** “Methods of solving multipoint boundary value problems for systems of loaded differential equations”
   4. The total amount of the project is 5 000 000 (five million) tenge, including with a breakdown by years, for implementing the works according to item 3:

* for 2020 – 3 000 000 (three million) tenge;
* for 2021 – 2 000 000 (two million) tenge.

1. **Characteristics of scientific and technical products by qualification features and economic indicators**
   1. Direction of the work: loaded differential equations.
   2. Applications: applied mathematics, mathematical physics, mathematical biology.
   3. Final result:

* for 2020: An algorithm for finding the solution to linear two-point boundary value problem for essentially loaded ordinary differential equations will be constructed;
* for 2021: An algorithm for finding the solution to linear multipoint boundary value problem for essentially loaded ordinary differential equations will be constructed and conditions for its convergence will be established. The unique solvability of linear multipoint boundary value problems for essentially ordinary differential equations will be established. Numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations will be developed.

According to the requirements of the call documentation, the following documents will be published:

- at least 1 (one) article or review published in a peer-reviewed scientific publication on the scientific direction of the project, included in 1 (first), 2 (second) or 3 (third) quartiles in the Web of Science database and (or) having a percentile according to CiteScore at least 50 (fifty) in the Scopus database, or printed in the indicated publications;

- as well as at least 1 (one) article in a peer-reviewed foreign and (or) domestic publication with a non-zero impact factor (recommended by CQAEA).

* 1. Patentability: not patentable.
  2. Scientific and technical level (novelty): high.
  3. The use of scientific and technical products is carried out: by the Executor.
  4. Type of use of the result of scientific and (or) scientific and technical activities: theoretical.

1. **Name of work, terms of their implementation and results**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Stage code | Name of works under the Contract and the main stages of its implementation | | | Deadlines | | | Expected results |
| Start | | Ending |
| **2020** | | | | | | | |
| 1.1 | Construction of an algorithm for finding a solution to linear two-point boundary value problems for essentially loaded ordinary differential equations | | | October  2020 | | December 2020 | An algorithm for finding the solution to linear two-point boundary value problem for essentially loaded ordinary differential equations will be constructed |
| **2021** | | | | | | | |
| 2.1 | | Construction of an algorithm for finding a solution to linear multipoint boundary value problems for essentially loaded ordinary differential equations and establishment of conditions for its convergence | January 2021 | | March 2021 | | An algorithm for finding the solution to linear multipoint boundary value problem for essentially loaded ordinary differential equations will be constructed and conditions for its convergence will be established |
| 2.2 | | Establishment the unique solvability of linear multipoint boundary value problems for essentially loaded ordinary differential equations | April 2021 | | June 2021 | | The unique solvability of linear multipoint boundary value problems for essentially ordinary differential equations will be established |
| 2.3 | | Development of numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations | July 2021 | | September 2021 | | Numerical methods for solving linear multipoint boundary value problems for essentially loaded ordinary differential equations will be developed.  Based on the results of the implementation of this scientific project for the entire period,:   * at least 1 (one) article or review published in a peer-reviewed scientific publication on the scientific direction of the project, included in 1 (first), 2 (second) or 3 (third) quartiles in the Web of Science database and (or) having a percentile according to CiteScore at least 50 (fifty) in the Scopus database, or printed in the indicated publications; * as well as at least 1 (one) article in a peer-reviewed foreign and (or) domestic publication with a non-zero impact factor (recommended by CQAEA). |

|  |  |
| --- | --- |
| From customer:  Chairman of the State Institution "Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan"  \_\_\_\_\_\_\_\_\_\_\_\_\_\_ Kurmangalieva Zh.D.  м.п. | From the Contractor:  General Director of the RSE on the RK "Institute of Mathematics and Mathematical Modeling" of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan  \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Sadybekov M.A.  м.п.  Familiarized with:  Scientific supervisor of the project  \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Kadirbayeva Zh.M. |